Lezione 5

Derivata esterna

M^n varietà $\qquad \omega \in \Omega^k(M) \qquad 0 \leq k \leq n$

Definiamo $\quad d\omega \in \Omega^{k+1}(M)$ in questo modo:

In carte: $\qquad \omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1, \dots, i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

MULTIINDICE $\quad I = \{i_1, \dots, i_k\}$
t.c. $1 \leq i_1 < \dots < i_k \leq n$

in $\mathbb{R}^n$: $\qquad \omega = \sum_I f_I \, dx^I \qquad dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d\omega = \sum_I df_I \wedge dx^I \quad (k+1)\text{-forma su } \mathbb{R}^n$$

È ben definita?

<u>Prop</u>: Sì. Inoltre valgono queste proprietà:

1) $d: \Omega^k(M) \to \Omega^{k+1}(M)$ è lineare

2) $\omega \in \Omega^k(M)$ $\quad \eta \in \Omega^h(M)$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

3) $d \circ d = 0$ cioè

$$d(d\omega) = 0 \quad \forall \omega \in \Omega^k(M)$$

4) Se $f \in \mathcal{C}^\infty(M) = \Omega^0(M)$ allora $df \in \Omega^1(M)$ è l'usuale differenziale

<u>dim</u>: Dimostro 1), 2), 3), 4) in carte

4) OK $\quad$ 1) OK

2) $\omega = f\,dx^I \qquad \eta = g\,dx^J$

$|I| = k \qquad |J| = h$

$$d(\omega \wedge \eta) = d\left(f\,dx^I \wedge g\,dx^J\right)$$

$$= d\left(fg\,dx^I \wedge dx^J\right)$$

(per def) $= d(fg) \wedge dx^I \wedge dx^J$

$$= (df)\cdot g \wedge dx^I \wedge dx^J + f\cdot(dg) \wedge dx^I \wedge dx^J$$

$$= (df \wedge dx^I) \wedge (g\,dx^J) +$$

$$(-1)^k (f\,dx^I) \wedge (dg \wedge dx^J) = d\omega \wedge \eta +$$

$$+ (-1)^k \omega \wedge \eta$$

3) $d(d\omega) = 0$

$\omega = f\,dx^J$

$$d\omega = df \wedge dx^J = \frac{\partial f}{\partial x^i} dx^i \wedge dx^J$$

$$d(d\omega) = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i \wedge dx^J$$

$$= \underbrace{\frac{\partial^2 f}{\partial x^j \partial x^i}}_{\text{simmetrico in } i,j} \underbrace{dx^j \wedge dx^i}_{\text{antisimmetrico in } i,j} \wedge dx^J = 0$$

Buona def? Le proprietà 1) 2) 3) 4) caratterizzano completamente d in carte

$$\omega = \sum f_I dx^I$$

1) $\Rightarrow$ $d\omega$ dipende solo da $d(f_I dx^I)$

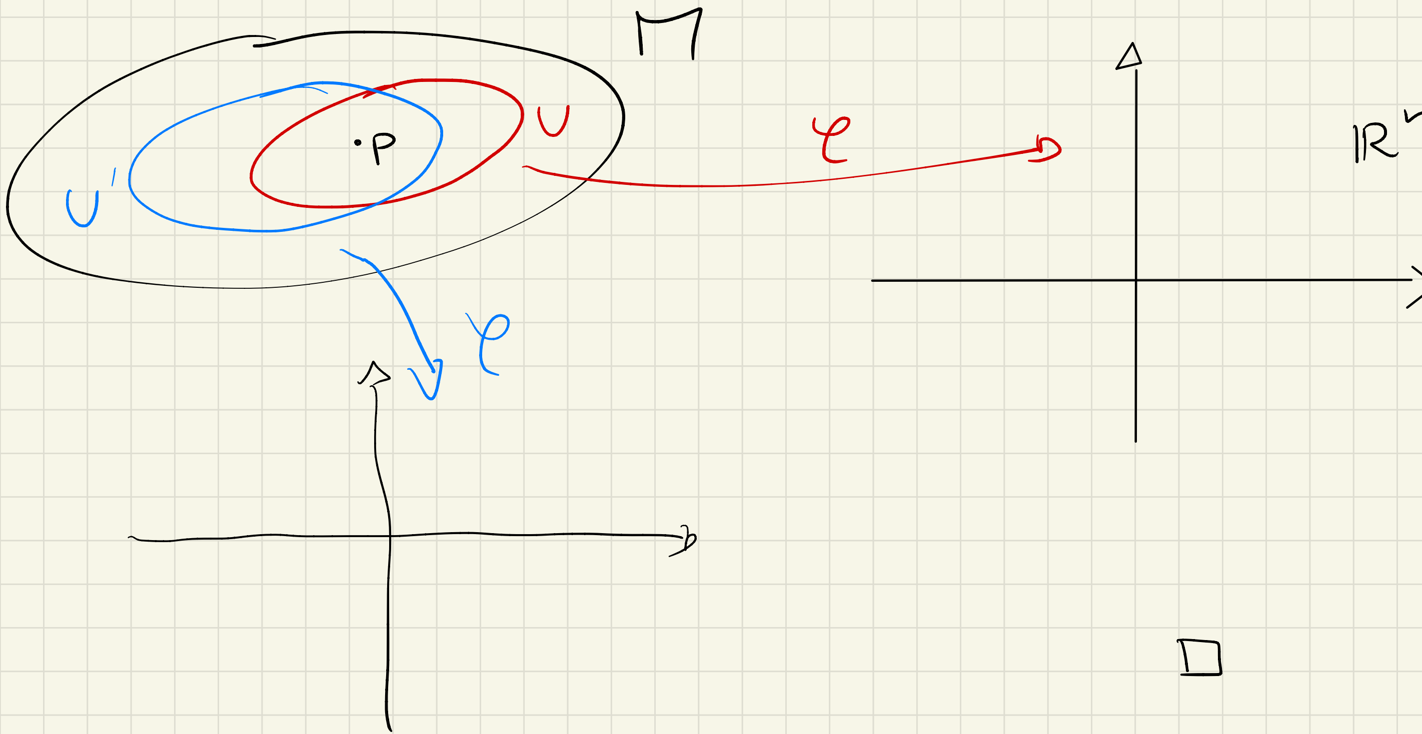
2) $d(f_I dx^I) \underset{(2)}{=} df_I \wedge dx^I + f_I \wedge d(dx^I)$

(4) è lui

$$d(dx^I) = d(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = 0$$

(3) e (2) Esercizio

Otteniamo buona def dall'unicità



$\square$

Esempi in $\mathbb{R}^3$:

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\| & & \cong & & \cong & & \cong \\
C^\infty(\mathbb{R}^3) & \xrightarrow[\nabla]{} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow[\text{rot}]{} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow[\text{div}]{} & C^\infty(\mathbb{R}^3)
\end{array}$$

esercizio esercizio

$\omega = f_1 dx^1 + \dots + f_n dx^n$

Posso interpretare ω come campo vettoriale $X = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

$$\boxed{\mathbb{R}^n \xrightarrow{\sim} (\mathbb{R}^n)^*}$$

$V \qquad V^*$

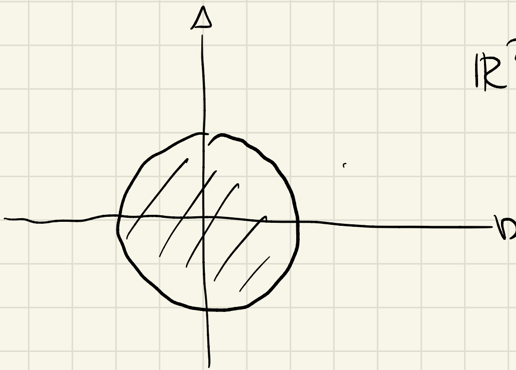
$$\omega = f_1 dx^2 \wedge dx^3 + f_2 dx^3 \wedge dx^1 + f_3 dx^1 \wedge dx^2$$

$$\omega = f\, dx^1 \wedge dx^2 \wedge dx^3$$

$df = \frac{\partial f}{\partial x^i} dx^i$ $\qquad$ $d \circ d = 0 \Rightarrow \mathrm{rot} \circ \nabla = 0$

$\mathrm{div} \circ \mathrm{rot} = 0$

VARIETÀ CON BORDO



$\mathbb{R}^2$

D^n DISCO

$D^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$

non è n-varietà

$B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$

BALL

B^n è n-varietà

Fatto generale: Qualsiasi aperto $U \subseteq M^n$ aperto è n-varietà

Atlante: $\mathcal{A} = \{\varphi_i : U_i \to V_i\}$ di M

$\mathcal{A}_U = \{\varphi_i|_U : U_i \cap U \to V_i \cap \varphi_i(U)\}$

Def: Una n-VARIETÀ CON BORDO è

M sp. top. (Hausdorff & base numerabile)

con $\mathcal{A} = \{ \varphi_i : U_i \to V_i \subseteq \mathbb{R}^n_+ \}$ atlante liscio

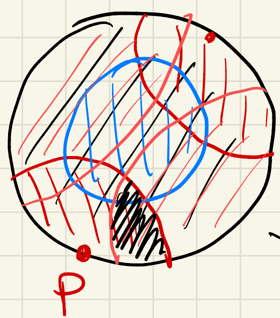
unica modifica

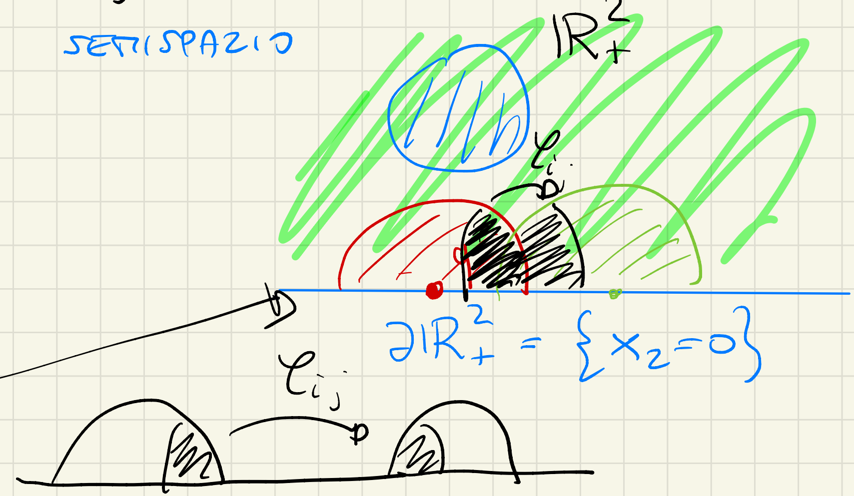
φ_{ij} lisce

$$\mathbb{R}^n_+ = \{ (x_1, \ldots, x_n) \mid x_n \geq 0 \}$$

SEMISPAZIO

$$\partial \mathbb{R}^n_+ = \{ x_n = 0 \} \text{ iperpiano}$$



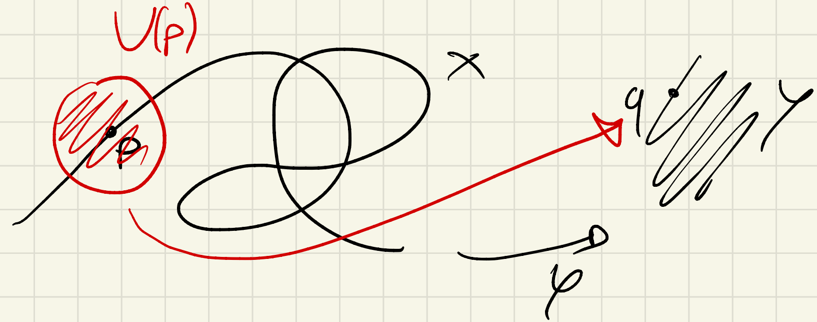


Def: $X \subseteq \mathbb{R}^k$ $\quad Y \subseteq \mathbb{R}^h$ qualsiasi

$\varphi : X \to Y$ è liscia

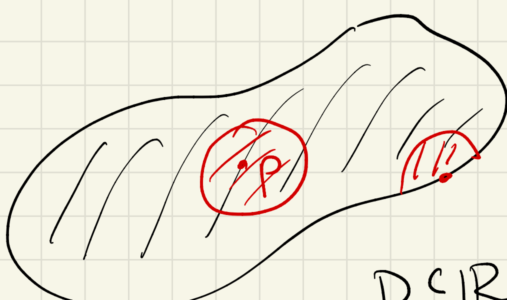
se è

loc. estendibile a funzioni lisce



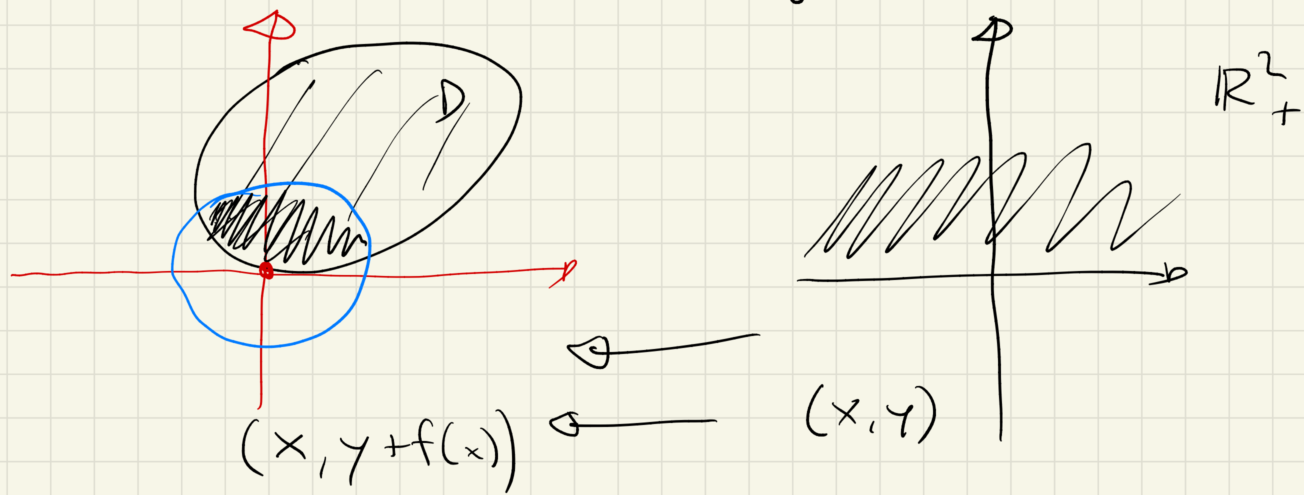
Es: Un DOMINIO REGOLARE in $\mathbb{R}^n$

$D =$



$D \subseteq \mathbb{R}^n$ t.c. $\forall p \in D \ \exists \ U(p) \to V \subseteq \mathbb{R}^n_+$ aperto

Es: $D^n \subseteq \mathbb{R}^n$ è dominio regolare



M^n varietà con bordo

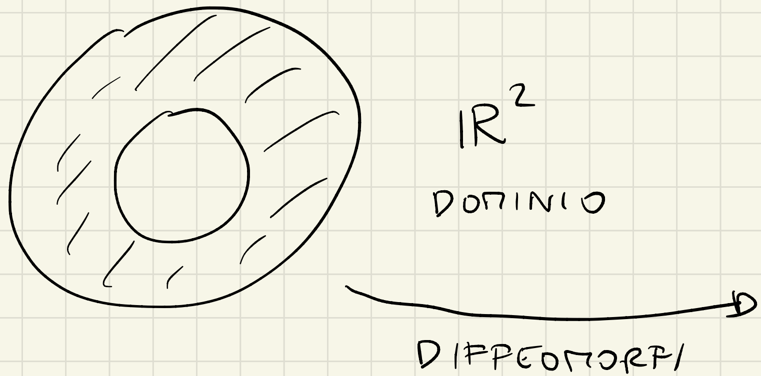
Def: $\partial M^n := \{ p \in M$ che vengono mandati in $\partial \mathbb{R}^n_+$ da qualche carta $\}$

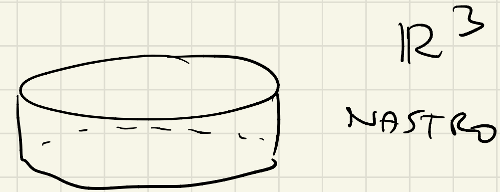
BORDO DI M

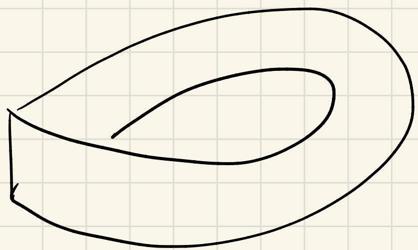
$p \in M \setminus \partial M$ è detto INTERNO

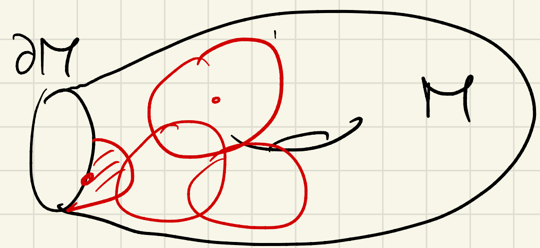
Es:

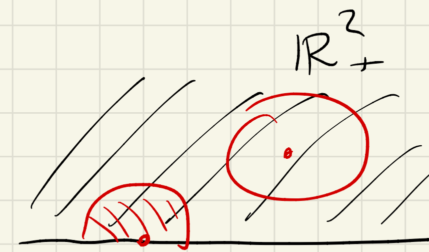
Superfici con bordo in $\mathbb{R}^3$











Oss: ∂M^n è una $(n-1)$-varietà senza bordo

Basta restringere l'atlante con codominio

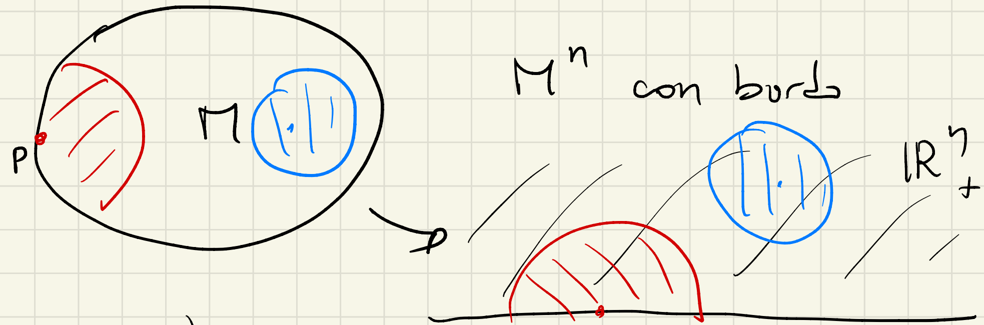
$$\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$$

$$\partial(\partial M^n) = \emptyset \qquad\qquad \omega \in \Omega^k(M)$$

$$d(d\omega) = 0$$

Gli spazi tangenti:

M^n con bordo



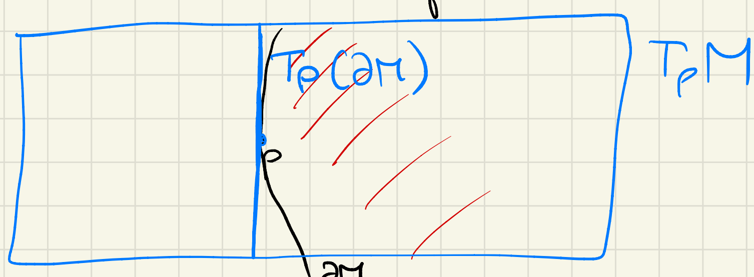
È definito $T_p M$

$\forall p \in M$

come prima (con le derivazioni)

NON con le curve!

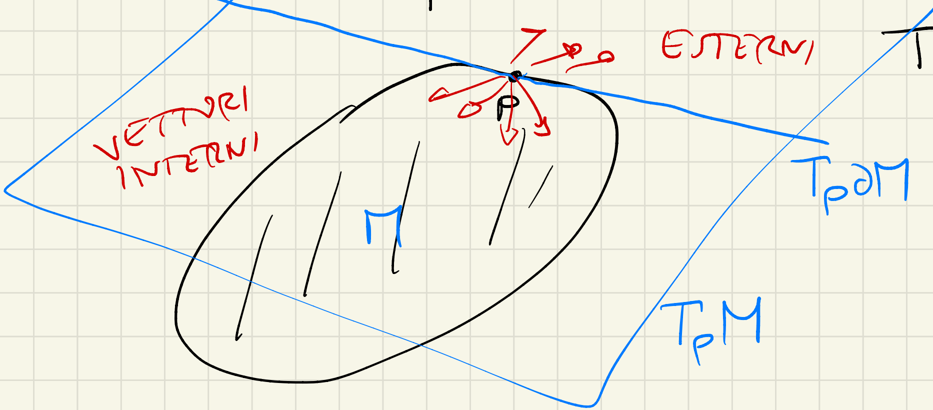
È sempre vero che $\dim T_p M = \dim M$



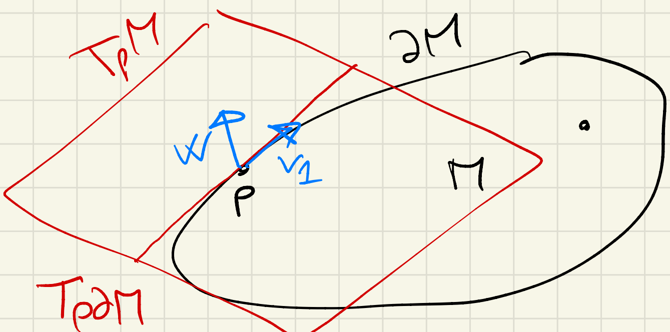
$p \in \partial M \subseteq M$

$T_p(\partial M)$ ha dim $n-1$

$T_p(M)$ " " n



<u>Orientazione</u>: Una ORIENTAZIONE per M^n ne induce una sul bordo ∂M ← dim $n-1$

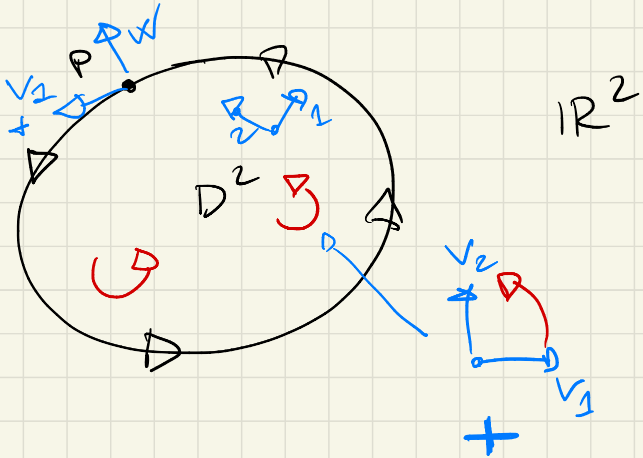


$p \in \partial M$

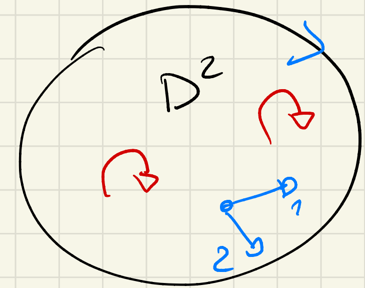
<u>Def</u>: $v_1, \dots, v_{n-1} \in T_p \partial M$ base è POSITIVA se $w, v_1, \dots, v_{n-1}$ è base positiva di T_pM

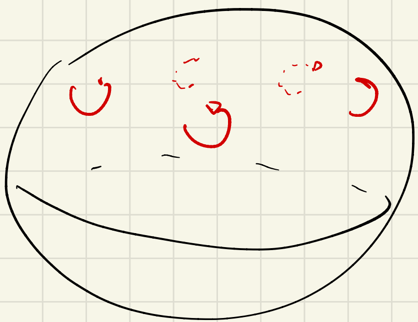
↑ esterno qualsiasi

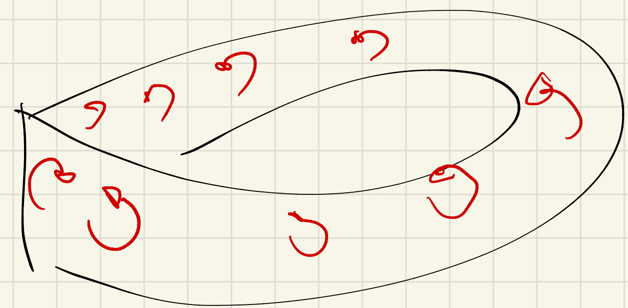
Esempi:

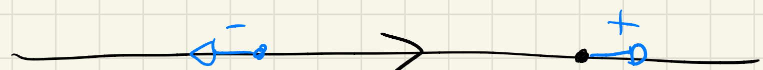


OUTWARD FIRST



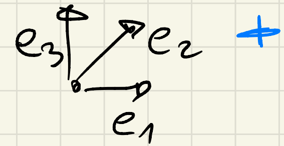


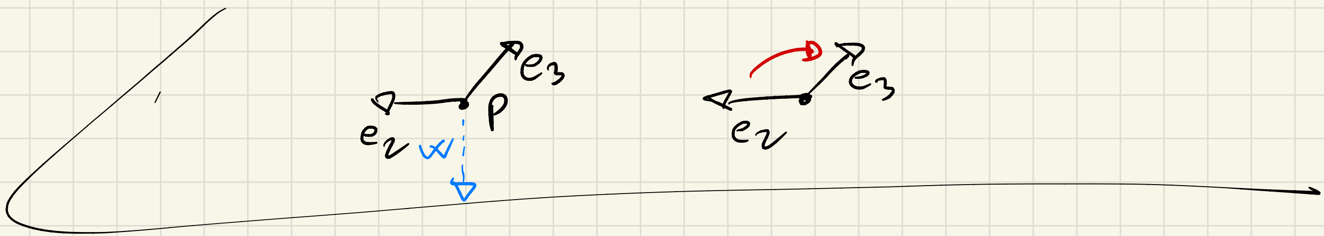




$\mathbb{R}^3_+ = \{z \geq 0\}$

semispazio





$\mathbb{R}^n_+$ orientato da $e_1, \dots, e_n$

$\partial \mathbb{R}^n_+$ orientazione indotta

$\quad \|$

$\mathbb{R}^{n-1}$ orientato da $e_1, \dots, e_{n-1}$

è la stessa solo per n pari